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The tangential map and associated integrable equations

V E Adler

L D Landau Institute for Theoretical Physics, 1a Ak. Semenov pr., Chernogolovka 142432, Russia

E-mail: adler@itp.ac.ru

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Abstract
The tangential map is a map on the set of smooth planar curves. It satisfies the 3D-consistency property and is closely related to some well-known integrable equations.

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1. Introduction

Let smooth planar curves C, C_1 and C_2 be given. Draw the tangent line to C through any point r on this curve, and let it meet C_1 in a point r_1 and C_2 in a point r_2. Let the tangent lines to the respective curves through these points meet at a point r_{12}. When the point r moves along C, the point r_{12} draws a new curve C_{12}. Thus, a local mapping on the set of planar curves is defined:

$$F : (C, C_1, C_2) \mapsto C_{12},$$

which will be referred to as the *tangential map*. The word 'local' means that, first, the mapping is defined not for all triples of curves since the tangent to the curve C may not intersect C_1 or C_2, and therefore only such curves or parts of the curves are considered where the construction is possible; second, the mapping may be multivalued since there may be several intersections, in this case a fixed branch of the mapping is considered.

Some properties of the tangential map are studied in this article. It turns out to be rather simply related to the factorization of differential operators. In its turn, this allows us to establish a relation with such integrable equations as the semidiscrete ($\Delta\Delta D$) Toda lattice and, under a reduction, the Hirota equation ($\Delta\Delta$). One of the modifications of the discrete Kadomtsev–Petviashvili (KP) equation ($\Delta\Delta\Delta$) appears in the discrete version of the tangential map. Thus, the tangential mapping is not a quite new object, rather it is of certain interest as one more geometric interpretation of well-known integrable equations.

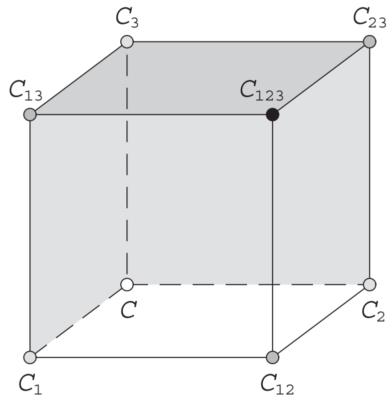

Figure 1. 3D-consistency or consistency around a cube. The white vertices correspond to the given curves; the shaded faces show one of three possible construction ways of the curve corresponding to the black vertex.

2. 3D-consistency

The main property of the tangential map is 3D-consistency. This means that if one starts from the curves C, C_1, C_2, C_3 and constructs the curves $C_{ij} = F(C, C_i, C_j)$ then the curve C_{123} constructed from the triple C_i, C_{ij}, C_{ik} is one and the same for any permutation of i, j, k. Alternatively, this can be formulated as follows.

Theorem 1. *The tangential map satisfies the (local) identity,*

$$\begin{aligned} C_{123} &= F(C_1, F(C, C_1, C_2), F(C, C_1, C_3)) \\ &= F(C_2, F(C, C_1, C_2), F(C, C_2, C_3)) \\ &= F(C_3, F(C, C_1, C_3), F(C, C_2, C_3)). \end{aligned} \tag{1}$$

The proof is given in the following section. The combinatorial structure of identity (1) is represented by assigning the arguments of the mapping (the curves in our case) to the vertices of a cube, and the mapping itself to the faces, as shown in figure 1. The *N*-fold iteration of the mapping is associated with an $(N + 1)$-dimensional cube.

The notion of 3D-consistency was formulated in [1, 2] in connection to the discrete integrable equations of the difference KdV-type (the fields in the vertices of the cube) or to the Yang–Baxter-type mappings (the fields on the edges of the cube). Both types of equation appear as a nonlinear superposition principle for Darboux–Bäcklund transformations and are two dimensional: two discrete independent variables correspond to the shifts along the edges of an elementary square. In contrast, the case of the tangential map is related to a three-dimensional equation: in addition to the discrete variables, a continuous one appears corresponding to a parameter along the curves. Another important distinction is the asymmetry of the tangential map: the roles of the involved curves are obviously different. In particular, the formulae from the following section make clear that the construction of C_{ij} from C, C_i, C_j is described by a differential rational mapping, while the construction, for instance, of C_j from C, C_i, C_{ij} requires an additional integration. This explains the choice of the set $C, C_i, C_j, C_k, \ldots$ as preferable initial data, rather than the sequence $C, C_i, C_{ij}, C_{ijk}, \ldots$, as is usual in the standard formulation of Yang–Baxter mappings [3].

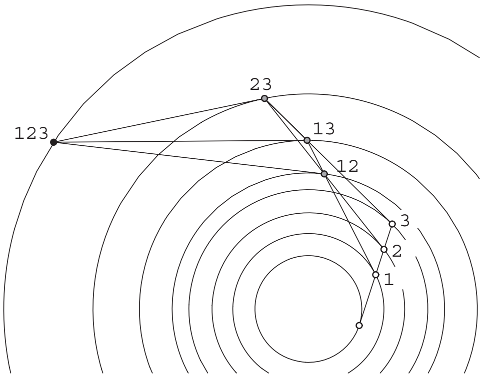

Figure 2. 3D-consistency of the tangential map in the simplest case of concentric circles.

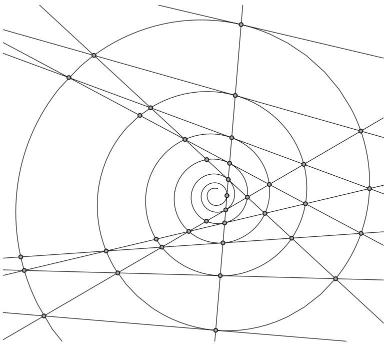

Figure 3. Jacob Bernoulli's wording 'eadem mutata resurgo' means that the logarithmic spiral is invariant with respect to a lot of geometrical transformations. The tangential map also preserves this curve.

The simplest example illustrating the tangential map and its 3D-consistency property is given by the family of concentric circles (see figure 2), or, in a slightly more general form, by the family of logarithmic spirals defined by equations $\rho_i = c_i \mathrm{e}^{\gamma\varphi}$ in polar coordinates. The fact that the tangential map does not lead out of these families is clear from the invariance of the construction with respect to the rotations and scalings. However, the concurrence of the last three tangents is not spontaneously obvious (as an additional feature, in the case of family of circles, the points r_{12}, r_{13}, r_{23} lie on the circle with Or_{123} as diameter, where O is the center of the family). Of course, this can be proved by elementary methods, but the point is that the concurrence of the tangents occurs in a much more general situation. In section 4.1, it will be demonstrated that this example corresponds to the simplest case of the factorization of differential operators with constant coefficients. Figure 3 illustrates a nice property of the logarithmic spiral (in contrast to the previous plot, this one contains only three generations of points r, r_i, r_{ij}, that is the triple intersections of the lines are not shown). This property can be formulated as the identity $F(C, C, C) = C$ for any branch of the map F.

Theorem 2. *Consider the intersection points of the logarithmic spiral with its tangent. The tangents through these points meet in the same spiral.*

3. Factorization of differential operators

Let the curve C be given in a parametric form $r = r(t)$. Then the point of intersection with the curve C_i is given, in the affine coordinates, by an equation of the form

$$r_i(t) = r(t) + a^i(t)\dot{r}(t), \qquad \dot{r} := D(r) := \frac{\mathrm{d}r}{\mathrm{d}t} \tag{2}$$

with some coefficient a^i (here and further on the superscript i marks the quantities associated with the edge CC_i of the combinatorial cube). This equation defines a parametrization of the curve C_i. It plays the role of an auxiliary linear problem for one branch of the tangential map. The curve C_{ij} is defined from the compatibility condition:

$$\begin{aligned} r_{ij} &= r_j + a^i_j\dot{r}_j = r + a^j\dot{r} + a^i_j D(r + a^j\dot{r}) \\ &= r_i + a^j_i\dot{r}_i = r + a^i\dot{r} + a^j_i D(r + a^i\dot{r}), \end{aligned}$$

where the coefficient a^i_j corresponds to the edge C_jC_{ij}.

Equating the coefficients for the linearly independent vectors $\dot{r}, \ddot{r}$ yields the equations,

$$a^j a^i_j = a^i a^j_i, \quad (1 + \dot{a}^j)a^i_j + a^j = (1 + \dot{a}^i)a^j_i + a^i, \tag{3}$$

which can be solved as the differential mapping $(a^i, a^j) \mapsto (a^i_j, a^j_i)$:

$$a^i_j = \frac{(a^i - a^j)a^i}{a^i - a^j + a^i\dot{a}^j - \dot{a}^i a^j}. \tag{4}$$

This is the formula which defines the action of the tangential map on the coefficients a. Alternatively, one can use the first of equations (3) in order to introduce the potential v accordingly to the formula $a^i = v/v_i$, then the second equation rewrites as the differential mapping $f : (v, v_i, v_j) \mapsto v_{ij}$:

$$v_{ij} = \frac{v_i v_j}{v} + \frac{\dot{v}_i v_j - v_i \dot{v}_j}{v_j - v_i}. \tag{5}$$

The property of 3D-consistency is formulated in terms of the variables a as the commutativity of the operators $T_i : a^j \to a^j_i$ which define the shift along the edges CC_i:

$$T_iT_j(a^k) = T_jT_i(a^k), \tag{6}$$

and in terms of the variables v as the identity of type (1):

$$\begin{aligned} v_{123} &= f(v_1, f(v, v_1, v_2), f(v, v_1, v_3)) \\ &= f(v_2, f(v, v_1, v_2), f(v, v_2, v_3)) \\ &= f(v_3, f(v, v_1, v_3), f(v, v_2, v_3)). \end{aligned} \tag{7}$$

Both identities can be proved straightforwardly, although the computation is rather tedious. It can be avoided by the following argument.

Proof of theorem 1. The above compatibility condition is equivalent to the equality

$$\big(1 + a^j_i D\big)(1 + a^i D) = \big(1 + a^i_j D\big)(1 + a^j D), \tag{8}$$

that is the definition of the tangential map amounts to the reconstruction of an ordinary second-order differential operator from its kernel, under the condition of a unitary constant term which is equivalent to affine normalization. Consider the differential operator,

$$A = \big(1 + T_i\big(a^k_j\big)D\big)\big(1 + a^j_i D\big)(1 + a^i D),$$

corresponding to one of the three possible ways of computing r_{ijk}. According to (8), A is divisible from the right not only by $1 + a^i D$, but also by $1 + a^j D$. Moreover, the two left factors of A can be rewritten, again according to (8), as $(1+T_i(a^j_k)D)(1+a^k_iD)$, that is operator A does not change under permutation of j and k. But this means that it is divisible from the right by $1+a^kD$ as well. Therefore, the kernel of A is invariant with respect to any permutation of indices. Since a differential operator is uniquely defined by its kernel (up to a scalar factor which is fixed here by the condition that the constant term is unitary), hence operator A itself is invariant with respect to the permutations. □

Now it is clear that an N-fold tangential map corresponds to a differential operator of order N divisible from the right by operators $1+a^iD$, $i=1,\dots,N$. This immediately leads to the Wronskian formula (for each of two components of r):

$$r_{1,2,\dots,N} = \frac{\det\begin{pmatrix} r & \varphi_1 & \varphi_2 & \dots & \varphi_N \\ \dot r & \dot\varphi_1 & \dot\varphi_2 & \dots & \dot\varphi_N \\ \vdots & \vdots & \vdots & & \vdots \\ D^N(r) & D^N(\varphi_1) & D^N(\varphi_2) & \dots & D^N(\varphi_N)\end{pmatrix}}{\det\begin{pmatrix} \dot\varphi_1 & \dot\varphi_2 & \dots & \dot\varphi_N \\ \vdots & \vdots & & \vdots \\ D^N(\varphi_1) & D^N(\varphi_2) & \dots & D^N(\varphi_N)\end{pmatrix}}$$

where $a^i = -\varphi_i/\dot\varphi_i$.

Remark 1. More simple mappings of types (4) and (5) are obtained from the factorization of operators normalized by the condition of the unitary leading term:

$$\big(D-a^j_i\big)(D-a_i) = \big(D-a^i_j\big)(D-a_j),$$

which is equivalent to

$$(T_i-1)(a^j) = (T_j-1)(a^i), \qquad \dot a^i - \dot a^j = a_i a^j_i - a^j a^i_j,$$

and brings to the maps

$$a^i_j = a^i + \frac{\dot a^i - \dot a^j}{a^i - a^j} \tag{9}$$

and (under the substitution $a^i = v_i - v$)

$$v_{ij} = v_i + v_j - v + \frac{\dot v_i - \dot v_j}{v_i - v_j}. \tag{10}$$

The 3D-consistency of these maps is proved analogously.

Clearly, equations (4), (5) and (9), (10) are interpreted as three-dimensional equations on $\mathbb{Z}^2\times\mathbb{R}$, with the fields a corresponding to the edges of the lattice and v corresponding to the vertices. These equations are related via simple substitutions to the semidiscrete Toda lattice, introduced in [4] for the first time (to the best of author's knowledge); see also [5].

4. Examples and reductions

4.1. Logarithmic spirals

It is convenient to use the complex notation in this example, assuming $r=\mathrm{e}^{(\gamma+\mathrm{i})t}$ (the case $\gamma=0$ corresponds to the circle). Then $r_k = r + a^k\dot r = (1+\gamma a^k + \mathrm{i}a^k)\mathrm{e}^{(\gamma+\mathrm{i})t}$ and these curves are homothetic to the original one if and only if the coefficients a^k are constant. The action

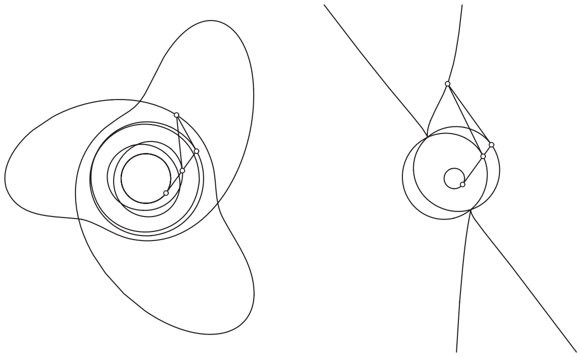

Figure 4. Examples of the tangential map.

of the map (4) on the constant coefficients is identical: $a^k_j = a^k$, therefore, the tangential map amounts to the rotational dilation:

$$r_{jk} = (1 + \gamma a^j + \mathrm{i}a^j)(1 + \gamma a^k + \mathrm{i}a^k)r$$

which preserves the family of curves under consideration. The N-fold mapping is given by an analogous explicit formula, so that this example can be considered trivial. However, even this example demonstrates that the established relation between the tangential map and differential operators is not one to one, and depends on the choice of the initial curve and its parametrization. The mappings corresponding to the same operators (that is, with the same coefficients a) can be regarded as locally equivalent, but the global picture may be quite different. For example, the tangential map has four branches in the case of concentric circles (real if the radius of C is less than the radii of C_1, C_2) and an infinite number of branches in the case of logarithmic spirals.

In order to obtain the auto-mapping shown in figure 3, one has to assume the additional constraint $r_k(t) = r(t + \delta_k)$ (then $r_{jk}(t) = r(t + \delta_j + \delta_k), r_{jkl}(t) = r(t + \delta_j + \delta_k + \delta_l)$ and so on), that is

$$\mathrm{e}^{(\gamma+\mathrm{i})(t+\delta_k)} = (1 + \gamma a^k + \mathrm{i}a^k)\mathrm{e}^{(\gamma+\mathrm{i})t}.$$

This implies that δ_k are roots of the transcendental equation,

$$\cos\delta - \gamma\sin\delta = \exp(-\gamma\delta),$$

and the coefficients a^k are expressed by the formula,

$$a^k = \exp(\gamma\delta_k)\sin\delta_k.$$

It can be derived from here (this is left to the reader as an easy exercise) that the boundary of the domain free of the lines in figure 3 is approximated by a parabola. The numeric values for this plot are $\gamma = 0.1, \delta_1 = 5.24, \delta_2 = 7.25, \ldots$.

4.2. Periodic coefficients

The previous example suggests that a picture with good global behavior of the curves can be obtained if the starting curve is the circle $r = \mathrm{e}^{\mathrm{i}t}$ again, and the coefficients $a^k(t)$ are almost constant functions with periods commensurable with π. For instance, the left plot in figure 4 corresponds to

$$a^1 = 1 + \frac{1}{5}\cos\frac{3}{2}t, \qquad a^2 = 2 + \frac{1}{10}\cos\left(\frac{t}{2} + \frac{\pi}{4}\right).$$

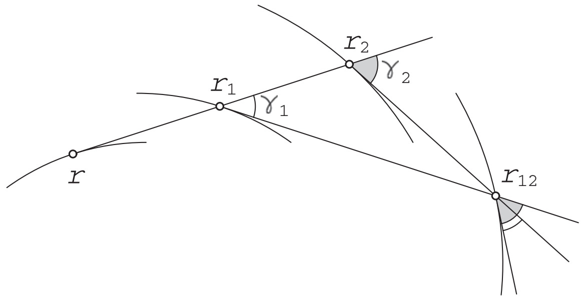

Figure 5. Loxodromic reduction of the tangential map.

The right plot corresponds to the choice

$$a^1 = 4 + \sin t, \quad a^2 = 4 + \cos t.$$

Here the tangential map brings to the curve with cusps.

4.3. Loxodromes

We will say, slightly abusing the terminology, that a curve $\widetilde{C}$ is a *loxodrome* for a given curve C if it intersects the tangents to C under a constant angle γ (in particular, if $\gamma = \pi/2$, then $\widetilde{C}$ is an involute of C).

Theorem 3 demonstrates that the tangential map preserves this type of relation between the curves (see figure 5). As a preliminary, it is convenient to introduce the parameter on the curve C according to equations,

$$\dot r = y(t)\tau, \quad \dot\tau = \nu, \quad \dot\nu = -\tau, \tag{11}$$

where τ, ν are unit tangent and normal vectors. Obviously, the function y is the radius of curvature and the relation to the natural parametrization by arc length $r = r(s)$ is

$$y(t(s)) = 1/\kappa(s), \quad \mathrm{d}t = \kappa(s)\mathrm{d}s.$$

This choice of parametrization is explained by the fact that its form is preserved for the curve $\widetilde{C}$ as well. Indeed, the equation of a point on this curve is $\tilde r = r + a\dot r = r + ay\tau$. Then

$$\dot{\tilde r} = (y + D(ay))\tau + ay\nu,$$

and since $\tilde r$ meets τ under the constant angle γ (reckoned in the direction of the normal ν, for definiteness), hence

$$y + D(ay) = ay \ \cot\ \gamma. \tag{12}$$

In virtue of this constraint, the equalities

$$\dot{\tilde r} = \tilde y\tilde\tau, \quad \dot{\tilde\tau} = \tilde\nu, \quad \dot{\tilde\nu} = -\tilde\tau,$$

hold, with

$$\tilde y = \frac{ay}{\sin\gamma}, \quad \tilde\tau = \tau\cos\gamma + \nu\sin\gamma, \quad \tilde\nu = \nu\cos\gamma - \tau\sin\gamma.$$

Thus, we have proved that the choice of t as the parameter brings to equations of the type (11) for the loxodrome $\widetilde{C}$ as well. If the function y is given then the constraint (12) is the determining equation for the coefficient a, and the loxodrome is constructed by any solution of it.

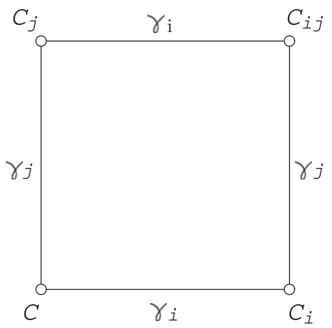

Figure 6. Bianchi diagram.

Theorem 3. *Let curves C_i and C_j meet tangents to a curve C under constant angles γ^i and γ^j, respectively, and $\gamma^i \neq \gamma^j$. Then the curve $C_{ij} = F(C, C_i, C_j)$ meets tangents to C_i under the angle γ^j and tangents to C_j under the angle γ^i.*

Proof. Assume the parametrization (11) for the curve C. Then, as was shown before, the parameters a^k of the tangential map and the functions y_k for the loxodromes are related by

$$y + D(a^k y) = a^k y \cot \gamma^k, \qquad y_k = \frac{a^k y}{\sin \gamma^k}, \qquad k = i, j.$$

It is easy to prove that in virtue of these constraints the tangential map (4) takes the form

$$a^i_j = \frac{a^i/a^j - 1}{\cot \gamma^j - \cot \gamma^i}$$

and, moreover, the identity

$$y_j + D\big(a^i_j y_j\big) = a^i_j y_j \cot \gamma^i$$

holds, which is the constraint of form (12) for the functions y_j and a^i_j. But this means exactly that the curve C_{ij} is a loxodrome for C_j, corresponding to the angle γ^i. □

Formally, this statement looks like the well-known Bianchi theorem on the permutability of Darboux–Bäcklund-type transformations (see figure 6). However, it is clear from the proof that in our case the situation is more simple: indeed, Bäcklund transformations amount to solving of Riccati equations, while the construction of the loxodromes amounts, according to (12), to a simple quadrature. The superposition principle for the transformations under consideration turns out to be linear:

$$\sin(\gamma^i - \gamma^j) y_{ij} = \sin(\gamma^i) y_i - \sin(\gamma^j) y_j.$$

A genuine Darboux transformation leading to the nonlinear superposition principle is provided by the reduction presented in the following example.

4.4. Darboux transformation

Let H_γ be the homothety with a coefficient $\gamma \neq 1$ with respect to a fixed point (origin) in the plane. We will say that the curves $C, \widetilde{C}$ are in the *tangential correspondence* with the parameter γ if the tangent to the curve C through any point r meets $H_\gamma(\widetilde{C})$ in the point $H_\gamma(\tilde{r})$, and the tangent to $\widetilde{C}$ through $\tilde{r}$ meets $H_\gamma(C)$ in the point $H_\gamma(r)$ (see figure 7).

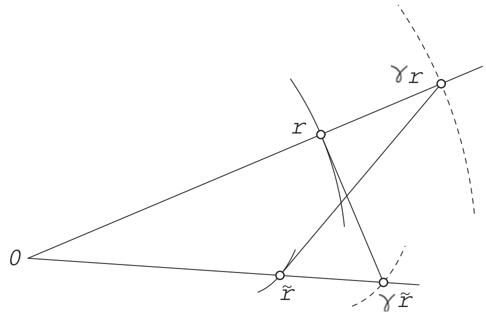

Figure 7. The tangential correspondence.

As in the example with the loxodromes, this notion defines some reduction of the tangential map which can be studied more conveniently in some special parametrization of the curves. The points on the curves $C, \widetilde{C}$ are related by equations of the form

$$\gamma \tilde r(t) = r(t) + a(t)\dot r(t), \quad \gamma r(t) = \tilde r(t) + b(t)\dot{\tilde r}(t),$$

which imply that the vector functions $r(t), \tilde r(t)$ satisfy linear second-order ODEs:

$$ab\ddot r + (\dot a b + a + b)\dot r + (1-\gamma^2)r = 0,$$
$$ab\ddot{\tilde r} + (a\dot b + a + b)\dot{\tilde r} + (1-\gamma^2)\tilde r = 0.$$

Therefore, the ratio of the first and the last coefficient is the same for r and $\tilde r$, that is it is an invariant of the tangential correspondence. It is convenient to use such a parametrization that this ratio is constant. Let

$$\lambda ab = 1-\gamma^2, \quad \lambda = (1-\gamma^2)\mu,$$

then the equations take the form

$$\ddot r + u\dot r + \lambda r = 0, \quad \ddot{\tilde r} + \tilde u\dot{\tilde r} + \lambda \tilde r = 0, \tag{13}$$

where functions $u, \tilde u$ are related to the coefficient a via the pair of Riccati equations:

$$\dot a + 1 - ua + \mu a^2 = 0, \quad -\dot a + 1 - \tilde u a + \mu a^2 = 0.$$

The first one is linearized by the substitution $a = -\phi/\dot\phi$ which brings to the equation $\ddot\phi + u\dot\phi + \mu\phi = 0$. Thus, the curve $\widetilde{C}$ is constructed by use of a particular scalar solution of the differential equation for the original curve C, at the value of the parameter $\lambda = \mu$, that is the tangential correspondence is nothing but an example of Darboux transformation.

Theorem 4. *Let a curve C be in the tangential correspondence with curves C_i, C_j, with parameters γ^i, γ^j, respectively, and $\gamma^i \neq \gamma^j$. Then a unique curve C_{ij} exists which is in the tangential correspondence with curves C_i, C_j, with parameters γ^j, γ^i, respectively. Moreover,*

$$H_{\gamma^i\gamma^j}(C_{ij}) = F(C, H_{\gamma^i}(C_i), H_{\gamma^j}(C_j)).$$

Proof. It follows directly from the definitions of the tangential map and the tangential correspondence that if such a curve C_{ij} exists then it is unique and is given by the above formula. So we only need to verify that this curve is indeed in the tangential correspondence with C_i, C_j, by use of the relations

$$\dot a^k + 1 = ua^k - \mu^k(a^k)^2, \quad u_k = -u + 2\mu^k a^k + \frac{2}{a^k}, \quad k = i, j.$$

It is easy to prove that formula (4) takes, in virtue of these constraints, the algebraic form

$$a^i_j = \frac{a^i - a^j}{a^j(\mu^i a^i - \mu^j a^j)}, \tag{14}$$

and that the coefficient $A = a^i_j$ identically satisfies the Riccati equation:

$$\dot{A} + 1 - u_j A + \mu^i A^2 = 0.$$

This means that the curves C_{ij}, C_j are in tangential correspondence, with the parameter γ^i. □

It should be noted that equation (13) for r defines the spectral problem for the sinh-Gordon equation, mapping (14) is equivalent to mapping (F_4) from [6], and the substitution $a^i = v/v_i$, $u = 2\dot{v}/v$ brings to the equation

$$v_{ij}(v_j - v_i) = v(\mu_i v_j - \mu_j v_i)$$

which is equivalent to the Hirota equation [7].

Remark 2. Analogously, mapping (9) admits the similar reduction:

$$\dot{a}^k = u + \gamma^k - (a^k)^2, \qquad u_k = -u - 2\gamma^k + 2(a^k)^2,$$

which brings to the mapping

$$a^i_j = -a^j + \frac{\gamma^i - \gamma^j}{a^i - a^j}.$$

This is one of the forms of the nonlinear superposition principle of Darboux transformation for the Shrödinger operator [8].

5. Further generalizations

5.1. Discrete tangential map

Let us consider discrete curves $r = r(n)$, then an analog of equation (2) reads

$$r_i = r + a^i(T-1)(r), \qquad T : n \mapsto n+1.$$

The compatibility condition of such equations,

$$\begin{aligned} r_{ij} &= \big(1 + a^i_j(T-1)\big)(1 + a^j(T-1))(r) \\ &= \big(1 + a^j_i(T-1)\big)(1 + a^i(T-1))(r), \end{aligned}$$

brings to the relations

$$T(a^j)a^i_j = T(a^i)a^j_i, \qquad (1 - a^j)a^i_j + a^j = (1 - a^i)a^j_i + a^i,$$

which yield a discrete analog of the tangential map (4):

$$T_j(a^i) = a^i_j = \frac{(a^i - a^j)T(a^i)}{(1 - a^j)T(a^i) - (1 - a^i)T(a^j)}. \tag{15}$$

The substitution $a^i = T(v)/v_i$ brings to an analog of map (5):

$$f : (v, v_i, v_j) \mapsto v_{ij} = \frac{v_i v_j T(v_j - v_i)}{T(v)(v_j - v_i)} + \frac{T(v_i)v_j - v_i T(v_j)}{v_j - v_i}. \tag{16}$$

The symmetric form of this equation,

$$\frac{T(v_j - v_i)}{T(v)} + \frac{T(v_i) - v_{ij}}{v_i} + \frac{v_{ij} - T(v_j)}{v_j} = 0,$$

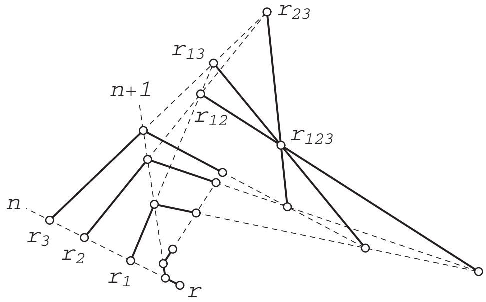

Figure 8. 3D-consistency of the discrete tangential map.

demonstrates that the shift T is actually on the equal footing with T_i and T_j. Difference substitutions relate this equation to the discrete Toda and KP equations (in particular, the variable v is identified as the wavefunction of the linear problem for the KP equation [9]). Alternative geometric interpretations of these equations can be found in papers [9, 10]. More precisely, the geometry of the discrete tangential map is essentially the same as of the Menelaus lattice [9], since the points $r_i, r_j, r_{ij}, T(r), T(r_i)$ and $T(r_j)$ are the vertices of the complete quadrilateral governed by the Menelaus theorem; see figure 8.

The 3D-consistency property of maps (15) and (16) is formulated by the same general identities (6), (7) as in the continuous case, and is proved along the lines of the proof of theorem 1. There exists also the simple geometric explanation of this property[1]: the triangles $r_{12}(n)r_{13}(n)r_{23}(n)$ and $r_{12}(n+1)r_{13}(n+1)r_{23}(n+1)$ are in perspective with respect to the line $r(n+1)r(n+2)$ (marked by $n+1$ in figure 8); therefore, according to Desargues theorem, the lines $r_{12}(n)r_{12}(n+1)$, $r_{13}(n)r_{13}(n+1)$ and $r_{23}(n)r_{23}(n+1)$ are concurrent, as required. It is clear that the 3D-consistency of the original tangential maps (4) and (5) may be derived from here once again via the natural continuous limit.

It should be noted that papers [9, 10] contain no indication of the multidimensional consistency property. In contrast to the more generic but logically simpler case of quadrilateral lattices [11], the understanding of the consistency for equations of type (16) is a quite recent achievement which is not explicitly presented in the literature yet (see the forthcoming paper [12] where a general theory of this class of equations is developed).

5.2. The higher-order maps

The tangential map can be defined for the curves in a space of any dimension by the same equation (2) which means that the curves C_i should be taken on the ruled surface generated by the tangents to the base curve C. A more general possibility is to consider the osculating subspaces instead of the tangents. Namely, equation (2) can be replaced by

$$r_i(t) = r(t) + a^{1,i}(t)\dot{r}(t) + \cdots + a^{m,i}(t)D^m(r(t)),$$

for the curves in a space of dimension greater or equal to $2m$. In the discrete case, one has analogously

$$r_i(n) = r(n) + a^{1,i}(n)(r(n+1) - r(n)) + \cdots + a^{m,i}(n)(r(n+m) - r(n)).$$

[1] This proof is due to WK Schief.

It is not difficult to demonstrate that the compatibility condition, $T_jT_i(r) = T_iT_j(r)$, is equivalent to a system of $2m$ equations which can be solved in the form of a differential or difference mapping:

$$\left(a^{1,i},\ldots,a^{m,i},a^{1,j},\ldots,a^{m,j}\right)\mapsto\left(a_j^{1,i},\ldots,a_j^{m,i},a_i^{1,j},\ldots,a_i^{m,j}\right),$$

with the derivatives or shifts up to mth order. These mappings are 3D-consistent, which can be proved by arguments analogous to the case $m = 1$ just considered. Unfortunately, the explicit form of these mappings is too bulky already at $m = 2$, so it would be desirable to find some reduction lowering their order and/or number of fields.

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References

[1] Nijhoff F W and Walker A J 2001 The discrete and continuous Painlevé hierarchy and the Garnier system *Glasgow Math. J.* A **43** 109–23
[2] Bobenko A I and Suris Yu B 2002 Integrable systems on quad-graphs *Int. Math. Res. Not.* **2002** 573–611
[3] Veselov A P 2003 Yang-Baxter maps and integrable dynamics *Phys. Lett.* A **314** 214–21
[4] Levi D, Pilloni L and Santini P M 1981 Integrable three-dimensional lattices *J. Phys. A: Math. Gen.* **14** 1567–75
[5] Adler V E and Startsev S Ya 1999 Discrete analogues of the Liouville equation *Theor. Math. Phys.* **121** 271–84
[6] Adler V E, Bobenko A I and Suris Yu B 2004 Geometry of Yang–Baxter maps: pencils of conics and quadrirational mappings *Commun. Anal. Geom.* **12** 967–1007
[7] Hirota R 1977 Nonlinear partial difference equations: III. Discrete sine-Gordon equation *J. Phys. Soc. Japan* **43** 2079–86
[8] Veselov A P and Shabat A B 1993 Dressing chain and the spectral theory of Schrödinger operators *Funct. Anal. Appl.* **27** 81–96
[9] Konopelchenko B G and Schief W K 2002 Menelaus' theorem, Clifford configurations and inversive geometry of the Schwarzian KP hierarchy *J. Phys. A: Math. Gen.* **35** 6125–44
[10] Doliwa A 1997 Geometric discretization of the Toda system *Phys. Lett.* A **234** 187–92
[11] Doliwa A and Santini P M 1997 Multidimensional quadrilateral lattices are integrable *Phys. Lett.* A **233** 365–72
[12] Adler V E, Bobenko A I and Suris Yu B 2009 The classification of integrable discrete equations of octahedron type, forthcoming